TITU ANDREESCU DORIN ANDRICA

360 Problems for Mathematical Contests

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FOREWORD

I take great pleasure in recommending to all readers - Romanians or from abroad - the book of professors Titu Andreescu and Dorin Andrica. This book is the fruit of a prodigious activity of the two authors, well-known creators of mathematics questions for Olympiads and other mathematical contests. They have published innumerable original problems in various mathematical journals.

The book is organized in six chapters: algebra, number theory, geometry, trigonometry, analysis and comprehensive problems. In addition, other fields of mathematics found their place in this book, for example, combinatorial problems can be found in the last chapter, and problems involving complex numbers are included in the trigonometry section. Moreover, in all chapters of this book the serious reader can find numerous challenging inequality problems. All featured problems are interesting, with an increased level of difficulty; some of them are real gems that will give great satisfaction to any math lover attempting to solve or even extend them.

Through their outstanding work as jury members of the National Mathematical Olympiad, the Balkan Mathematics Contest (BMO), and the International Mathematical Olympiad (IMO), the authors also supported the excellent results of the Romanian contestants in these competitions. A great effort was given in preparing lectures for summer and winter training camps and also for creating original problems to be used in selection tests to search for truly gifted mathematics students. To support the claim that the Romanian students selected to represent the country were really the ones to deserve such honor, we note that only two mathematicians of Romanian origin, both former IMO gold-medalists, were invited recently to give conferences at the International Mathematical Congress: Dan Voiculescu (Zürich, 1994) and Daniel Tataru (Beijing, 2002). The Romanian mathematical community unanimously recognized this outstanding activity of professors Titu Andreescu and Dorin Andrica. As a consequence, Titu Andreescu, at that time professor at Loga Academy in Timişoara and having students on the team participating in the IMO, was appointed to serve as deputy leader of the national team. Nowadays, Titu's potential, as with other Romanians in different fields, has been fully realized in the United States, leading the USA team in the IMO, coordinating the training and selection of team contestants and serving as member of several national and regional mathematical contest juries.

One more time, I strongly express my belief that the 360 mathematics problems featured in this book will reveal the beauty of mathematics to all students and it will be a guide to their teachers and professors.

Professor Ioan Tomescu
Department of Mathematics and Computer Science
University of Bucharest
Associate member of the Romanian Academy

FROM THE AUTHORS

This book is intended to help students preparing for all rounds of Mathematical Olympiads or any other significant mathematics contest. Teachers will also find this work useful in training young talented students.

Our experience as contestants was a great asset in preparing this book. To this we added our vast personal experience from the other side of the "barricade", as creators of problems and members of numerous contest committees.

All the featured problems are supposed to be original. They are the fruit of our collaboration for the last 30 years with several elementary mathematics journals from all over the world. Many of these problems were used in contests throughout these years, from the first round to the international level. It is possible that some problems are already known, but this is not critical. The important thing is that an educated - to a certain extent - reader will find in this book problems that bring something new and will teach new ways of dealing with key mathematics concepts, a variety of methods, tactics, and strategies.

The problems are divided in chapters, although this division is not firm, for some of the problems require background in several fields of mathematics.

Besides the traditional fields: algebra, geometry, trigonometry and analysis, we devoted an entire chapter to number theory, because many contest problems require knowledge in this field.

The comprehensive problems in the last chapter are also intended to help undergraduate students participating in mathematics contests hone their problem solving skills. Students and teachers can find here ideas and questions that can be interesting topics for mathematics circles.

Due to the difficulty level of the problems contained in this book, we deemed it appropriate to give a very clear and complete presentation of all solutions. In many cases, alternative solutions are provided.

As a piece of advice to all readers, we suggest that they try to find their own solutions to the problems before reading the given ones. Many problems can be solved in multiple ways and pertain to interesting extensions.

This edition is significantly different from the 2002 Romanian edition. It features more recent problems, enhanced solutions, along with references for all published problems.

We wish to extend our gratitude to everyone who influenced in one way or another the final version of this book.

We will gladly receive any observation from the readers.

The authors

Chapter 1
ALGEBRA

PROBLEMS

1. Let C be a set of n characters $\{c_1, c_2, \ldots, c_n\}$. We call **word** a string of at most m characters, $m \leq n$, that does not start nor end with c_1.

How many **words** can be formed with the characters of the set C?

2. The numbers $1, 2, \ldots, 5n$ are divided into two disjoint sets. Prove that these sets contain at least n pairs (x, y), $x > y$, such that the number $x - y$ is also an element of the set which contains the pair.

3. Let $a_1, a_2, \ldots, a_n$ be distinct numbers from the interval $[a, b]$ and let σ be a permutation of $\{1, 2, \ldots, n\}$.

Define the function $f : [a, b] \to [a, b]$ as follows:

$$f(x) = \begin{cases} a_{\sigma(i)} & \text{if } x = a_i,\ i = \overline{1, n} \\ x & \text{otherwise} \end{cases}$$

Prove that there is a positive integer h such that $f^{[h]}(x) = x$, where $f^{[h]} = \underbrace{f \circ f \circ \cdots \circ f}_{h \text{ times}}$.

4. Prove that if x, y, z are nonzero real numbers with $x + y + z = 0$, then

$$\frac{x^2 + y^2}{x + y} + \frac{y^2 + z^2}{y + z} + \frac{z^2 + x^2}{z + x} = \frac{x^3}{yz} + \frac{y^3}{zx} + \frac{z^3}{xy}.$$

5. Let a, b, c, d be complex numbers with $a + b + c + d = 0$. Prove that

$$a^3 + b^3 + c^3 + d^3 = 3(abc + bcd + cda + dab).$$

6. Let a, b, c be nonzero real numbers such that $a + b + c = 0$ and $a^3 + b^3 + c^3 = a^5 + b^5 + c^5$. Prove that

$$a^2 + b^2 + c^2 = \frac{6}{5}$$

7. Let a, b, c, d be integers. Prove that $a + b + c + d$ divides

$$2(a^4 + b^4 + c^4 + d^4) - (a^2 + b^2 + c^2 + d^2)^2 + 8abcd.$$

8. Solve in complex numbers the equation

$$(x+1)(x+2)(x+3)^2(x+4)(x+5) = 360.$$

9. Solve in real numbers the equation

$$\sqrt{x} + \sqrt{y} + 2\sqrt{z-2} + \sqrt{u} + \sqrt{v} = x + y + z + u + v.$$

10. Find the real solutions to the equation

$$(x+y)^2 = (x+1)(y-1).$$

11. Solve the equation

$$\sqrt{x + \sqrt{4x + \sqrt{16x + \sqrt{\cdots + \sqrt{4^n x + 3}}}}} - \sqrt{x} = 1.$$

12. Solve the equation

$$\sqrt{x+a} + \sqrt{x+b} + \sqrt{x+c} = \sqrt{x+a+b-c},$$

where a, b, c are real parameters. Discuss the equation in terms of the values of the parameters.

13. Let a and b be distinct positive real numbers. Find all pairs of positive real numbers (x, y), solutions to the system of equations

$$\begin{cases} x^4 - y^4 = ax - by \\ x^2 - y^2 = \sqrt[3]{a^2 - b^2}. \end{cases}$$

14. Solve the equation

$$\left[\frac{25x-2}{4}\right] = \frac{13x+4}{3},$$

where $[a]$ denotes the integer part of a real number a.

15. Prove that if $a \geq \dfrac{1+\sqrt{5}}{2}$, then

$$\left[\frac{1 + \left[\frac{1+na^2}{a}\right]}{a}\right] = n, \quad n = 0, 1, 2, \ldots$$

16. Prove that if x, y, z are real numbers such that $x^3 + y^3 + z^3 \neq 0$, then the ratio

$$\frac{2xyz - (x+y+z)}{x^3 + y^3 + z^3}$$

equals $\frac{2}{3}$ if and only if $x + y + z = 0$.

17. Solve in real numbers the equation

$$\sqrt{x_1 - 1} + 2\sqrt{x_2 - 4} + \cdots + n\sqrt{x_n - n^2} = \frac{1}{2}(x_1 + x_2 + \cdots + x_n).$$

18. Find the real solutions to the system of equations

$$\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = 9 \\ \left(\dfrac{1}{\sqrt[3]{x}} + \dfrac{1}{\sqrt[3]{y}}\right)\left(1 + \dfrac{1}{\sqrt[3]{x}}\right)\left(1 + \dfrac{1}{\sqrt[3]{y}}\right) = 18 \end{cases}$$

19. Solve in real numbers the system of equations

$$\begin{cases} y^2 + u^2 + v^2 + w^2 = 4x - 1 \\ x^2 + u^2 + v^2 + w^2 = 4y - 1 \\ x^2 + y^2 + v^2 + w^2 = 4u - 1 \\ x^2 + y^2 + u^2 + w^2 = 4v - 1 \\ x^2 + y^2 + u^2 + v^2 = 4w - 1 \end{cases}$$

20. Let a_1, a_2, a_3, a_4, a_5 be real numbers such that $a_1 + a_2 + a_3 + a_4 + a_5 = 0$ and $\max\limits_{1 \le i < j \le 5} |a_i - a_j| \le 1$. Prove that $a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 \le 10$.

21. Let a, b, c be positive real numbers. Prove that

$$\frac{1}{2a} + \frac{1}{2b} + \frac{1}{2c} \ge \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}$$

22. Let a, b, c be real numbers such that the sum of any two of them is not equal to zero. Prove that

$$\frac{a^5 + b^5 + c^5 - (a+b+c)^5}{a^3 + b^3 + c^3 - (a+b+c)^3} \ge \frac{10}{9}(a+b+c)^2$$

23. Let a, b, c be real numbers such that $abc = 1$. Prove that at most two of the numbers

$$2a - \frac{1}{b}, \quad 2b - \frac{1}{c}, \quad 2c - \frac{1}{a}$$

are greater than 1.

24. Let a, b, c, d be real numbers. Prove that

$$\min(a - b^2, b - c^2, c - d^2, d - a^2) \leq \frac{1}{4}.$$

25. Let $a_1, a_2, \ldots, a_n$ be numbers in the interval $(0, 1)$ and let $k \geq 2$ be an integer. Find the maximum value of the expression

$$\sum_{i=1}^{n} \sqrt[k]{a_i(1 - a_{i+1})},$$

where $a_{n+1} = a_1$.

26. Let m and n be positive integers. Prove that

$$\frac{x^{mn} - 1}{m} \geq \frac{x^n - 1}{x}$$

for any positive real number x.

27. Prove that $m! \geq (n!)^{\left[\frac{m}{n}\right]}$ for all positive integers m and n.

28. Prove that

$$1 + \frac{1}{\sqrt[n]{2}} + \frac{1}{\sqrt[n]{3}} + \cdots + \frac{1}{\sqrt[n]{n}} > n \sqrt[n]{\frac{2}{n+1}}$$

for any integer $n \geq 2$.

29. Prove that

$$n\left(1 - 1/\sqrt[n]{n}\right) + 1 > 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} > n\left(\sqrt[n]{n+1} - 1\right)$$

for any positive integer n.

30. Let $a_1, a_2, \ldots, a_n \in (0, 1)$ and let $t_n = \dfrac{n a_1 a_2 \ldots a_n}{a_1 + a_2 + \cdots + a_n}$. Prove that

$$\sum_{i=1}^{n} \log_{a_i} t_n \geq (n - 1)n.$$

31. Prove that between n and $3n$ there is at least a perfect cube for any integer $n \geq 10$.

32. Compute the sum

$$S_n = \sum_{k=1}^{n} (-1)^{\frac{k(k+1)}{2}} k.$$

33. Compute the sums:
a) $S_n = \displaystyle\sum_{k=0}^{n} \frac{1}{(k+1)(k+2)} \binom{n}{k}$; b) $T_n = \displaystyle\sum_{k=0}^{n} \frac{1}{(k+1)(k+2)(k+3)} \binom{n}{k}$.

34. Show that for any positive integer n the number

$$\binom{2n+1}{0}2^{2n}+\binom{2n+1}{2}2^{2n-2}\cdot 3+\cdots+\binom{2n+1}{2n}3^n$$

is the sum of two consecutive perfect squares.

35. Evaluate the sums:

$$S_n=\binom{n}{1}-3\binom{n}{3}+5\binom{n}{5}-7\binom{n}{7}+\ldots$$

36. Prove that

$$1^2\binom{n}{1}+3^2\binom{n}{3}+5^2\binom{n}{5}+\cdots=n(n+1)2^{n-3}$$

for all integers $n\geq 3$.

37. Prove that

$$\sum_{k=1}^{2^n}[\log_2 k]=(n-2)2^n+n+2$$

for all positive integers n.

38. Let $x_n=2^{2^n}+1$, $n=1,2,3,\ldots$ Prove that

$$\frac{1}{x_1}+\frac{2}{x_2}+\frac{2^2}{x_3}+\cdots+\frac{2^{n-1}}{x_n}<\frac{1}{3}$$

for all positive integers n.

39. Let $f:\mathbb{C}\to\mathbb{C}$ be a function such that $f(z)f(iz)=z^2$ for all $z\in\mathbb{C}$. Prove that

$$f(z)+f(-z)=0 \text{ for all } z\in\mathbb{C}$$

40. Consider a function $f:(0,\infty)\to\mathbb{R}$ and a real number $a>0$ such that $f(a)=1$. Prove that if

$$f(x)f(y)+f\left(\frac{a}{x}\right)f\left(\frac{a}{y}\right)=2f(xy) \text{ for all } x,y\in(0,\infty),$$

then f is a constant function.

41. Find with proof if the function $f:\mathbb{R}\to[-1,1]$, $f(x)=\sin[x]$ is periodical.

42. For all $i,j=\overline{1,n}$ define $S(i,j)=\sum_{k=1}^{n}k^{i+j}$. Evaluate the determinant $\Delta=|S(i,j)|$.

43. Let

$$x_{ij} = \begin{cases} a_i & \text{if } \ i = j \\ 0 & \text{if } \ i \neq j,\ i + j \neq 2n + 1 \\ b_i & \text{if } \ i + j = 2n + 1 \end{cases}$$

where a_i, b_i are real numbers.

Evaluate the determinant $\Delta_{2n} = |x_{ij}|$.

44. a) Compute the determinant

$$\begin{vmatrix} x & y & z & v \\ y & x & v & z \\ z & v & x & y \\ v & z & y & x \end{vmatrix}$$

b) Prove that if the numbers $\overline{abcd}$, $\overline{badc}$, $\overline{cdab}$, $\overline{dcba}$ are divisible by a prime p, then at least one of the numbers

$$a + b + c + d, \quad a + b - c - d, \quad a - b + c - d, \quad a - b - c + d,$$

is divisible by p.

45. Consider the quadratic polynomials $t_1(x) = x^2 + p_1 x + q_1^2$ and $t_2(x) = x^2 + p_2 x + q_2^2$, where p_1, p_2, q_1, q_2 are real numbers.

Prove that if polynomials t_1 and t_2 have zeros of the same nature, then the polynomial

$$t(x) = x^2 + (p_1 p_2 + 4q_1 q_2)x + (p_1 q_2 + p_2 q_1)^2$$

has real zeros.

46. Let a, b, c be real numbers with $a > 0$ such that the quadratic polynomial

$$T(x) = ax^2 + bcx + b^3 + c^3 - 4abc$$

has nonreal zeros.

Prove that exactly one of the polynomials $T_1(x) = ax^2 + bx + c$ and $T_2(x) = ax^2 + cx + b$ has only positive values.

47. Consider the polynomials with complex coefficients

$$P(x) = x^n + a_1 x^{n-1} + \cdots + a_n \text{ and } Q(x) = x^n + b_1 x^{n-1} + \cdots + b_n$$

having zeros $x_1, x_2, \ldots, x_n$ and $x_1^2, x_2^2, \ldots, x_n^2$ respectively.

Prove that if $a_1 + a_3 + a_5 + \ldots$ and $a_2 + a_4 + a_6 + \ldots$ are real numbers, then $b_1 + b_2 + \cdots + b_n$ is also a real number.