Arkansas Tech University
MATH 2924: Calculus II
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50 Polar Coordinates

Up to this point we have dealt exclusively with the Cartesian coordinate system. However, as we will see, this is not always the easiest coordinate system to work in. In this section we will start looking at the polar coordinate system.
The Cartesian system consists of two rectangular axes. A point P in this system is uniquely determined by two points x and y as shown in Figure 50.1(a). The polar coordinate system consists of a point O, called the **pole,** and a half-axis starting at O and pointing to the right, known as the **polar axis**. A point P in this system is determined by two numbers: the distance r between P and O and an angle θ between the ray OP and the polar axis as shown in Figure 50.1(b).

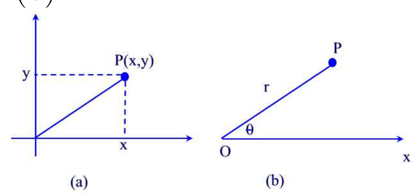

Figure 50.1

The Cartesian and polar coordinates can be combined into one figure as shown in Figure 50.2.

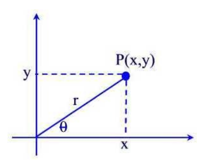

Figure 50.2

Figure 50.2 reveals the relationship between the Cartesian and polar coordinates:

$$r = \sqrt{x^2 + y^2} \qquad x = r\cos\theta \qquad y = r\sin\theta.$$

The following example illustrates the process of converting between polar and Cartesian coordinates.

Example 50.1
(a) Convert $\left(4, \frac{2\pi}{3}\right)$ to Cartesian coordinates.
(b) Convert $(-1, -1)$ into polar coordinates.

Solution.
(a) Plugging into the above formulas we find

$$\begin{aligned} x &= 4\cos\left(\frac{2\pi}{3}\right) = 4\left(-\frac{1}{2}\right) = -2 \\ y &= 4\sin\left(\frac{2\pi}{3}\right) = 4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3} \end{aligned}$$

So, in Cartesian coordinates this point is $(-2, 2\sqrt{3})$.
(b) Since $(-1,-1)$ is in the third quadrant, we have $r = \sqrt{2}, \cos\theta = -\frac{\sqrt{2}}{2}, \sin\theta = -\frac{\sqrt{2}}{2}$. Thus, $\theta = \frac{5\pi}{4}$. In this case the point could also be written in polar coordinates as $(\sqrt{2}, \frac{5\pi}{4})$ ■

Remark 50.1
1. In general $\theta \neq \arctan\left(\frac{y}{x}\right)$ since it is not possible to determine which quadrant θ is in from the value of $\tan\theta$ alone. For instance, in the presious example we have $\tan\theta = -\frac{1}{\sqrt{3}}$. This says that θ can be either in the second or fourth quadrant. Hence, when finding θ is advisable to use $x = r\cos\theta$ and $y = r\sin\theta$ for that purpose.
2. In the polar coordinate system each point cannot be defined uniquely. For example, a point with the angle θ will be the same as the point with angle $\theta + 2\pi$ when they have the same r values. However, we often choose $0 \leq \theta < 2\pi$.

Plotting Curves in Polar Coordinates
Several important types of graphs have equations that are simpler in polar form than in Cartesian form. For example, the polar equation of a circle having radius a and centered at the origin is simply $r = a$. In other words, polar coordinate system is useful in describing two dimensional regions that may be difficult to describe using Cartesian coordinates. For example, graphing the circle $x^2 + y^2 = a^2$ in Cartesian coordinates requires two functions, one for the upper half and one for the lower half. In polar coordinate system, the same circle has the very simple representation $r = a$.
The equation of a curve expressed in polar coordinates is known as a **polar equation,** and is usually written in the form $r = f(\theta)$. To construct graphs within the polar coordinate system you need to find how the value of r changes as θ changes. It is best to construct a table with certain values of θ and work out the respective value of r using the equation $r = f(\theta)$.

Remark 50.2
In our definition, r is positive. However, graphs of curves in polar coordinates

are traditionally drawn using negative values of r as well, because this makes the graphs symmetric.

Example 50.2 *(Polar Roses)*
A **polar rose** is a famous mathematical curve which looks like a petalled flower, and which can only be expressed as a polar equation. It is given by the equations $r = a \sin k\theta$ or $r = a \cos k\theta$ where $a > 0$ and k is an integer. These equations will produce a $k-$petalled rose if k is odd, or a $2k-$petalled rose if k is even. Graph the roses $r = 4 \sin 2\theta$ and $r = 4 \sin 3\theta$.

Solution.
The graph of $r = 4 \sin 2\theta$ is given in Figure 50.3(a) and the graph of $r = 4 \sin 3\theta$ is given in Figure 50.3(b) ■

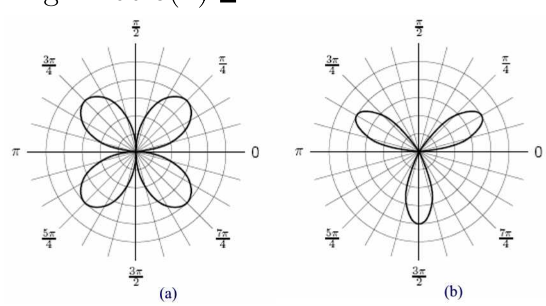

Figure 50.3

Example 50.3 *(Archemidean Spiral)*
The Archimedean spiral is a famous spiral that was discovered by Archimedes, which also can be expressed only as a polar equation. It is represented by the equation $r = a + b\theta$. Graph $r = \theta$.

Solution.
The graph of $r = \theta$ is given in Figure 50.4 ■

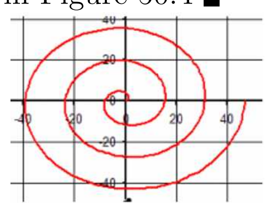

Figure 50.4

Example 50.4 *(Limaçons)*
Limaçons are curves with polar equations $r = b + a\cos\theta$ (horizontal limaçons) or $r = b + a\sin\theta$ (vertical limaçons) with $a, b > 0$. Graph $r = 1 + 2\cos\theta$ and $r = 3 + 2\cos\theta$.

Solution.
The graph of $r = 1+2\cos\theta$ is given in Figure 50.5(a) and that of $r = 3+2\cos\theta$ is given in Figure 50.5(b) ■

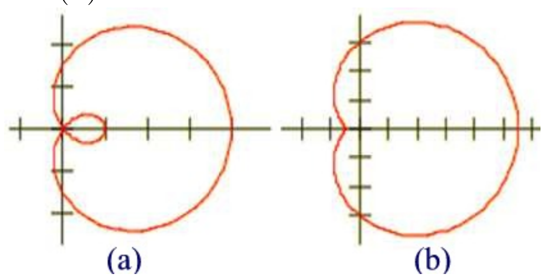

Figure 50.5

Example 50.5 *(Cardioids)*
Cardioids are curves with polar equations $r = a(1+\cos\theta)$ or $r = a(1+\sin\theta)$. Graph $r = 2(1+\cos\theta)$ and $r = 2(1+\sin\theta)$.

Solution.
The graph of $r = 2(1+\cos\theta)$ is given in Figure 50.6(a) and that of $r = 2(1+\sin\theta)$ is given in Figure 50.5 (b) ■

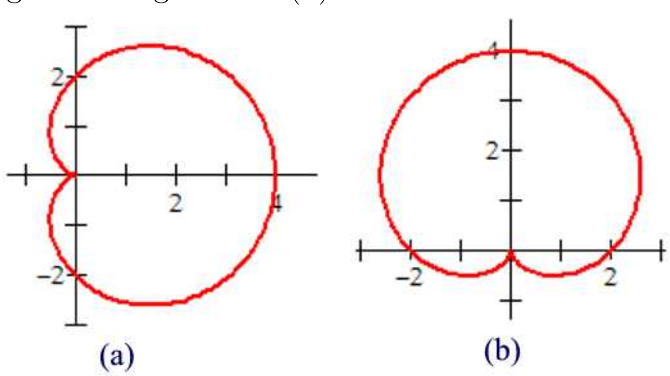

Figure 50.6

Area in Polar coordinates

Next we are going to look at areas enclosed by polar curves. Figure 50.7 shows a sketch of what the area that we will be finding in this section looks like.

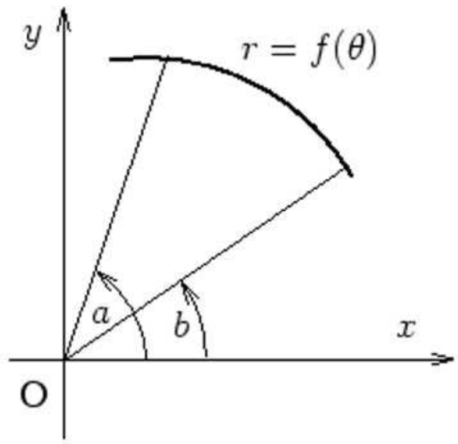

Figure 50.7

To find the area we use the "thin slices" approach. Divide the range of θ from a to b into lots of small parts of width $\Delta\theta$ and the area consequently being divided up into lots of thin pie slices. Any one of these slices, at angle θ, is approximately a triangle with sides $f(\theta)$ and and included angle $\Delta\theta$ as shown in Figure 50.8. The area of such a triangle is $\frac{1}{2}[f(\theta)]^2\sin(\Delta\theta)$. Since $\Delta\theta$ is assumed to be very small, $\sin(\Delta\theta)\approx\Delta\theta$. So we approximate the area of the pie-slice by $\frac{1}{2}r^2\Delta\theta$.

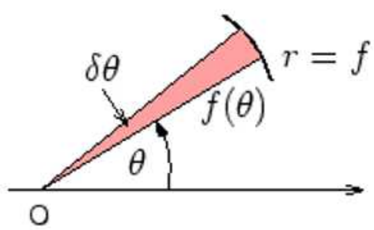

Figure 50.8

Adding up the thin slices to obtain

$$\text{Area of region} = \sum_{i=1}^{n}\frac{1}{2}r^2\Delta\theta$$

Taking the limit as $n \to \infty$ and $\Delta\theta \to 0$ we obtain

$$\text{Area of region} = \int_a^b \frac{1}{2} r^2 d\theta.$$

Example 50.6
Find the area inside the cardioid $r = a(1 + \cos\theta)$.

Solution.
Let A denote the area inside the cardioid. Using symmetry we have

$$\begin{array}{rcl} A & = & 2\int_0^\pi \frac{r^2}{2} d\theta \\ & = & \int_0^\pi a^2(1+\cos\theta)^2 d\theta \\ & = & a^2 \int_0^\theta (1 + 2\cos\theta + \cos^2\theta) d\theta \\ & = & a^2 \left[\theta + 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^\pi \\ & = & \frac{3a^2\pi}{2} \blacksquare \end{array}$$

Slope in Polar Coordinates
Since $x = r\cos\theta = f(\theta)\cos\theta$ and $y = f(\theta)\sin\theta$, by Section 22 we have

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}.$$

Example 50.7
Find the slope of the curve $r = \theta$ at the point $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Solution.
We are given that $x = \theta\cos\theta$ and $y = \theta\sin\theta$. Thus, $\frac{dx}{d\theta} = \cos\theta - \theta\sin\theta$ and $\frac{dy}{d\theta} = \sin\theta + \theta\cos\theta$. Thus,

$$\frac{dy}{dx}\Big|_{\theta=\frac{\pi}{2}} = \frac{\sin\theta + \theta\cos\theta}{\cos\theta - \theta\sin\theta}\Big|_{\theta=\frac{\pi}{2}} = -\frac{2}{\pi} \quad \blacksquare$$