Theoretical Problem 1

A A bungee jumper is attached to one end of a long elastic rope. The other end of the elastic rope is fixed to a high bridge. The jumper steps off the bridge and falls, from rest, towards the river below. He does not hit the water. The mass of the jumper is m, the unstretched length of the rope is L, the rope has a force constant (force to produce 1 m extension) of k and the gravitational field strength is g.

You may assume that

the jumper can be regarded as a point mass m attached to the end of the rope,

the mass of the rope is negligible compared to m,

the rope obeys Hooke's law,

air resistance can be ignored throughout the fall of the jumper.

Obtain expressions for the following and insert on the answer sheet:

(a) the distance y dropped by the jumper before coming instantaneously to rest for the first time,

(b) the maximum speed υ attained by the jumper during this drop,

(c) the time t taken during the drop before coming to rest for the first time.

B A heat engine operates between two identical bodies at different temperatures T_A and T_B ($T_A > T_B$), with each body having mass m and constant specific heat capacity s. The bodies remain at constant pressure and undergo no change of phase.

(a) Showing full working, obtain an expression for the final temperature T_0 attained by the two bodies A and B, if the heat engine extracts from the system the maximum amount of mechanical work that is theoretically possible.

Write your expression for the final temperature T_0 on the answer sheet.

(b) Hence, obtain and write on the answer sheet an expression for this maximum amount of work available.

The heat engine operates between two tanks of water each of volume 2.50 m^3. One tank is at 350 K and the other is at 300 K.

(c) Calculate the maximum amount of mechanical energy obtainable. Insert the value on the answer sheet.

Specific heat capacity of water = 4.19×10^3 J kg^{-1} K^{-1}

Density of water = 1.00×10^3 kg m^{-3}

C It is assumed that when the earth was formed the isotopes ^{238}U and ^{235}U were present but not their decay products. The decays of ^{238}U and ^{235}U are used to establish the age of the earth, *T*.

(a) The isotope ^{238}U decays with a half-life of 4.50×10^9 years. The decay products in the resulting radioactive series have half-lives short compared to this; to a first approximation their existence can be ignored. The decay series terminates in the stable lead isotope ^{206}Pb.

Obtain and insert on the answer sheet an expression for the number of ^{206}Pb atoms, denoted ^{206}n, produced by radioactive decay with time t, in terms of the present number of ^{238}U atoms, denoted ^{238}N, and the half-life time of ^{238}U. (You may find it helpful to work in units of 10^9 years.)

(b) Similarly, ^{235}U decays with a half-life of 0.710×10^9 years through a series of shorter half-life products to give the stable isotope ^{207}Pb.

Write down on the answer sheet an equation relating ^{207}n to ^{235}N and the half-life of ^{235}U.

(c) A uranium ore, mixed with a lead ore, is analysed with a mass spectrometer. The relative concentrations of the three lead isotopes ^{204}Pb, ^{206}Pb and ^{207}Pb are measured and the number of atoms are found to be in the ratios 1.00 : 29.6 : 22.6 respectively. The isotope ^{204}Pb is used for reference as it is not of radioactive origin. Analysing a pure lead ore gives ratios of 1.00 : 17.9 : 15.5.

Given that the ratio ^{238}N : ^{235}N is 137 : 1, derive and insert on the answer sheet an equation involving *T*.

(d) Assume that *T* is much greater than the half lives of both uranium isotopes and hence obtain an approximate value for *T*.

(e) This approximate value is clearly not significantly greater than the longer half life, but can be used to obtain a much more accurate value for *T*. Hence, or otherwise, estimate a value for the age of the earth correct to within 2%.

D Charge Q is uniformly distributed *in vacuo* throughout a spherical volume of radius R.

(a) Derive expressions for the electric field strength at distance r from the centre of the sphere for $r \leq R$ and $r > R$.

(b) Obtain an expression for the total electric energy associated with this distribution of charge.

Insert your answers to (a) and (b) on the answer sheet.

E A circular ring of thin copper wire is set rotating about a vertical diameter at a point within the Earth's magnetic field. The magnetic flux density of the Earth's magnetic field at this point is 44.5 μT directed at an angle of 64^{o} below the horizontal. Given that the density of copper is 8.90×10^{3} kg m^{-3} and its resistivity is 1.70×10^{-8} Ω m, calculate how long it will take for the angular velocity of the ring to halve. Show the steps of your working and insert the value of the time on the answer sheet. This time is much longer than the time for one revolution.

You may assume that the frictional effects of the supports and air are negligible, and for the purposes of this question you should ignore self-inductance effects, although these would not be negligible.

Question 1

A *Bungee Jumper*

(a) The jumper comes to rest when

lost gravitational potential energy = stored strain energy

$$mgy = \tfrac{1}{2} k (y-L)^2$$

0.1

$$ky^2 - 2y(kL + mg) + kL^2 = 0$$

0.1

This is solved as a quadratic.

$$y = \frac{2(kL+mg) \pm \sqrt{4(kL+mg)^2 - 4k^2L^2}}{2k}$$

$$= \frac{kL + mg \pm \sqrt{2mgkL + m^2g^2}}{k}$$

0.2

Need positive root; lower position of rest (other root after initial rise).

0.1

0.5

(b) The maximum speed is attained when the acceleration is zero and forces balance; i.e. when $mg = kx$ 0.1

Also kinetic energy = lost potential energy – strain energy within elastic rope

0.1

$$\tfrac{1}{2} m\upsilon^2 = mg(L+x) - \tfrac{1}{2} kx^2$$

0.1

$$x = \frac{mg}{k}$$

$$\upsilon^2 = 2g(L + \frac{mg}{k}) - \frac{mg^2}{k}$$

$$\upsilon = \sqrt{2gL + \frac{mg^2}{k}}$$

0.2

0.5

(c) Time to come to rest = time in free fall + time in SHM of rope to stop stretching

0.1

Length of free fall = $L = \tfrac{1}{2} g t_f^2$

Therefore $t_f = \sqrt{\frac{2L}{g}}$

0.2

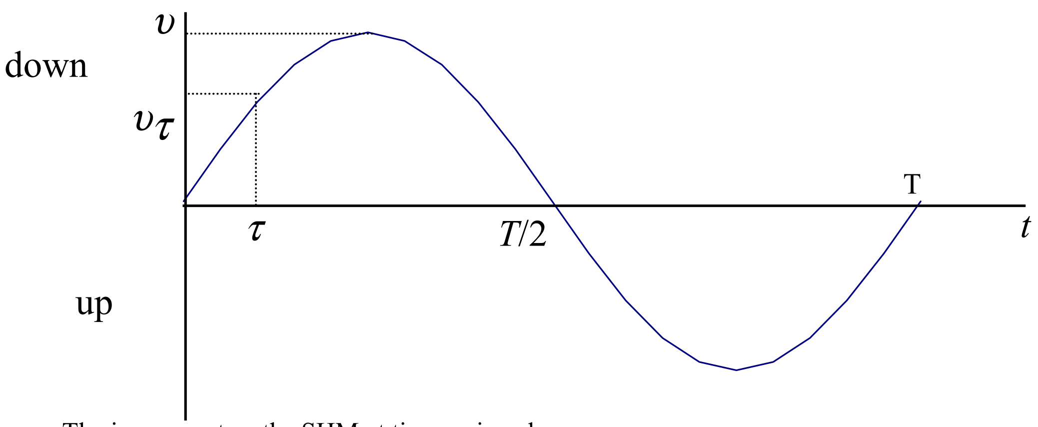
The jumper enters the SHM with free fall velocity $= gt_f = \sqrt{2gL} = \upsilon_\tau$

0.1

Period of SHM = $2\pi\sqrt{\frac{m}{k}} = T$

0.1

We represent a full SHM cycle by



The jumper enters the SHM at time τ given by

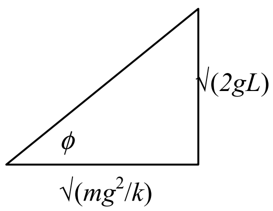
$$\tau = \frac{1}{\omega}\sin^{-1}\frac{\upsilon_\tau}{\upsilon} = \frac{1}{\omega}\sin^{-1}\frac{\sqrt{2gL}}{\upsilon}$$

0.2

Jumper comes to rest at one half cycle of the SHM at total time given by

$$= t_f + (T/2 - \tau)$$

0.1



$$= \sqrt{\frac{2L}{g}} + \pi\sqrt{\frac{m}{k}} - \frac{1}{\omega}\sin^{-1}\frac{\sqrt{2gL}}{\upsilon}$$

$$= \sqrt{\frac{2L}{g}} + \pi\sqrt{\frac{m}{k}} - \frac{1}{\omega}\sin^{-1}\frac{\sqrt{2gL}}{\sqrt{2gL + mg^2/k}}$$

$$= \sqrt{\frac{2L}{g}} + \sqrt{\frac{m}{k}}\ \{\pi -$$

$$\sin^{-1}\frac{\sqrt{2gL}}{\sqrt{2gL + mg^2/k}}\}$$

This is the same as

$$= \sqrt{\frac{2L}{g}} + \sqrt{\frac{m}{k}}\ \{\frac{\pi}{2} +$$

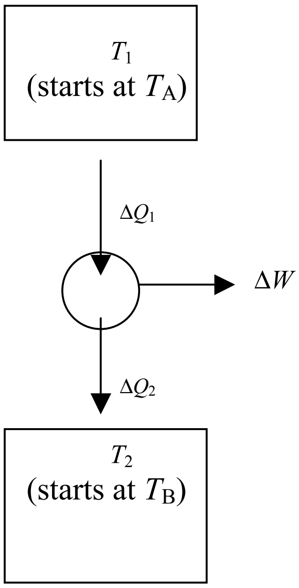
$$\cos^{-1}\frac{\sqrt{2gL}}{\sqrt{2gL + mg^2/k}}\}$$

$$= \sqrt{\frac{2L}{g}} + \sqrt{\frac{m}{k}}\tan^{-1}\{-\sqrt{\frac{2kL}{mg}}\}$$

0.2

1.0

B Heat Engine Question



In calculating work obtainable,
we assume no loss (friction etc.) in engine working.

ΔQ_1 = energy from body A
= $-ms\Delta T_1$ (ΔT_1 –ve)

$\Delta Q_2 = ms\Delta T_2$ (ΔT_2 +ve)

(a) For maximum amount of mechanical energy assume Carnot engine

$$\frac{\Delta Q_1}{T_1} = \frac{\Delta Q_2}{T_2}$$ throughout operation (second law)

0.2

But $\Delta Q_1 = -ms\Delta T_1$ and $\Delta Q_2 = ms\Delta T_2$

0.2

$$-ms\int_{T_A}^{T_0}\frac{dT_1}{T_1} = ms\int_{T_B}^{T_0}\frac{dT_2}{T_2}$$

0.1

$$\ln\frac{T_A}{T_0} = \ln\frac{T_0}{T_B}$$

0.1

$$T_0^2 = T_A T_B$$
$$T_0 = \sqrt{T_A T_B}$$

0.2

0.8

$$Q_1 = -ms \int_{T_A}^{T_0} \mathrm{d}T_1 = ms(T_A - T_0)$$

0.2

$$Q_2 = ms \int_{T_B}^{T_0} \mathrm{d}T_2 = ms(T_0 - T_B)$$

0.1

$$W = Q_1 - Q_2$$

0.2

$$W = ms(T_A - T_0 - T_0 + T_B) = ms(T_A + T_B - 2T_0) = ms\,(T_A + T_B - 2\sqrt{T_A T_B}\,)$$

or $ms(\sqrt{T_A} - \sqrt{T_B})^2$

0.2

0.7

(d) Numerical example:

Mass = volume × density

$$W = 2.50 \times 1.00 \times 10^3 \times 4.19 \times 10^3 \times (350 + 300 - 2\sqrt{350 \times 300}\,)\ \mathrm{J}$$
$$= 20 \times 10^6\ \mathrm{J}$$
$$= 20\ \mathrm{MJ}$$

0.5

0.5

C Radioactivity and age of the Earth

(a) $N = N_0 e^{-\lambda}$ N_0 = original number 0.1

$n = N_0(1 - e^{-\lambda t})$ 0.1

Therefore $n = N e^{\lambda t}(1 - e^{-\lambda t}) = N(e^{\lambda t} - 1)$ 0.1

So $n = N(2^{t/\tau} - 1)$ where τ is half-life

or as $\lambda = \dfrac{\ln 2}{T} = \dfrac{0.6931}{T}$, n = N($e^{\frac{0.6931t}{T}} - 1$) 0.1

$^{206}n = {}^{238}N(2^{t/4.50} - 1)$ or $^{206}n = {}^{238}N(e^{0.1540t} - 1)$ where time t is in 10^9 years 0.1

0.5

(b) $^{207}n = {}^{235}N(2^{t/0.710} - 1)$ or $^{207}n = {}^{235}N(e^{0.9762t} - 1)$ 0.1

0.1

(c) In mixed uranium (i.e. containing Pb of both natural and radioactive origin)

204 : 206 : 207 have proportions	1.00 : 29.6 : 22.6
In pure lead (no radioactivity)	1.00 : 17.9 : 15.5

Therefore for radioactively produced lead by subtraction

206 : 207 11.7 : 7.1 0.2

Dividing equations from (a) and (b) gives

$$\frac{^{206}n}{^{207}n} = \frac{^{238}N}{^{235}N}\{\frac{2^{t/4.50}-1}{2^{t/0.710}-1}\} \text{ or } \frac{^{206}n}{^{207}n} = \frac{^{238}N}{^{235}N}\{\frac{e^{0.1540t}-1}{e^{0.9762t}-1}\}$$

0.1

$$\frac{11.7}{7.1} = 137\{\frac{2^{T/4.50}-1}{2^{T/0.710}-1}\} \text{ or } \frac{11.7}{7.1} = 137\{\frac{e^{0.1540T}-1}{e^{0.9762T}-1}\}$$

0.1

$$0.0120\ \{2^{T/0.710} - 1\} = \{2^{T/4.50} - 1\}$$

$$\text{or } 0.0120\ \{e^{0.9762T} - 1\} = \{e^{0.1540T} - 1\}$$

0.1

0.5

(d) Assume $T >> 4.50 \times 10^9$ and ignore 1 in both brackets:

0.2

$$0.0120\ \{2^{T/0.710}\} = \{2^{T/4.50}\} \text{ or } 0.0120\ \{e^{0.9762T}\} = \{e^{0.1540T}\}$$

$$0.0120 = \{2^{T/4.50\ -\ T/0.710}\} = 2^{T(0.222-1.4084)} = 2^{-1.1862T}$$

$$T = -\frac{\log 0.0120}{\log 2 \times 1.1862} = 5.38$$

$T = 5.38 \times 10^9$ years

or $0.0120 = e^{-0.8222T}$ $\quad T = \dfrac{\ln 0.0120}{-0.8222} = \dfrac{-4.4228}{-0.8222} = 5.38$

$T = 5.38 \times 10^9$ years

0.2

0.4

(e) T is not >> 4.50×10^9 years but is > 0.71×10^9 years

0.1

We can insert the approximate value for T (call it $T^* = 5.38 \times 10^9$ years) in the $2^{T/4.50}$ term and obtain a better value by iteration in the rapidly changing $2^{T/0.710}$ term). We now leave in the –1's, although the –1 on the right-hand side has little effect and may be omitted).

0.1

Either $\quad 0.0120((2^{T/0.710} - 1) = 2^{T^*/4.50} - 1$

$$2^{T/0.710} - 1 = \frac{2^{1.1956} - 1}{0.0120} = \frac{2.2904 - 1}{0.0120} = 107.5$$

$$T = 0.710\frac{\log 108.5}{\log 2} = 4.80(0)$$

0.2

Put T* = 4.80(0) × 10^9 years

$$2^{T/0.710} = \frac{2^{1.0668} - 1}{0.0120} = \frac{2.0948 - 1}{0.0120} = 91.2$$

$$T = 0.710\frac{\log 91.2}{\log 2} = 4.62(3)$$

Further iteration gives 4.52

0.1

or

$0.0120(e^{0.9762T} - 1) = (e^{0.1540T^*} - 1)$ and similar

So more accurate answer for T to be in range 4.6 × 10^9 years to 4.5 × 10^9 years (either acceptable).

0.5

D Spherical charge

(a) Charge density = $\rho = \dfrac{Q}{\frac{4}{3}\pi R^3}$ within sphere

0.3

$x \le R$ Field at distance x:

$$E = \frac{\frac{4}{3}\pi x^3 \rho}{4\pi\varepsilon_0 x^2} = \frac{Qx}{4\pi\varepsilon_0 R^3}$$

0.3

$x > R$ Field at distance x from the centre: $E = \dfrac{Q}{4\pi\varepsilon_0 x^2}$

0.2
0.8

(b) *Method 1*

Energy density is $\frac{1}{2}\varepsilon_0 E^2$.

0.1

$x \le R$

Energy in a thin shell of thickness δx at radius x is given by

$$= \frac{1}{2}\varepsilon_0 E^2\ 4\pi x^2 \delta x = \frac{1}{2} 4\pi\varepsilon_0 \frac{Q^2 x^2}{(4\pi\varepsilon_0)^2 R^6} x^2\ \delta x$$ 0.1

Energy within the spherical volume = $\dfrac{1}{2}\dfrac{Q^2}{(4\pi\varepsilon_0)R^6}\displaystyle\int_{x=0}^{x=R} x^4 \mathrm{d}x = \dfrac{1}{40}\dfrac{Q^2}{\pi\varepsilon_0}\dfrac{1}{R}$

0.2

$x > R$

Energy within spherical shell = $\dfrac{1}{2}\varepsilon_0 E^2\ 4\pi x^2 \delta x = \dfrac{1}{2} 4\pi\varepsilon_0 \dfrac{Q^2}{(4\pi\varepsilon_0)^2 x^4} x^2\ \delta x$

0.1

Energy within the spherical volume for $x > R$

$$= \frac{1}{2}\frac{Q^2}{(4\pi\varepsilon_0)}\int_{x=R}^{x=\infty} \frac{1}{x^2}\mathrm{d}x = \frac{1}{8}\frac{Q^2}{\pi\varepsilon_0}\frac{1}{R}$$

0.2

Total energy associated with the charge distribution = $\dfrac{1}{40}\dfrac{Q^2}{\pi\varepsilon_0}\dfrac{1}{R}$

+ $\dfrac{1}{8}\dfrac{Q^2}{\pi\varepsilon_0}\dfrac{1}{R}$

$$= \frac{3}{20}\frac{Q^2}{\pi\varepsilon_0}\frac{1}{R}$$

0.1
0.8

Method 2

A shell with charge $4\pi x^2 \delta x \rho$ moves from ∞ to the surface of a sphere radius x where the electric potential is 0.1

$$\frac{\frac{4}{3}\pi x^3 \rho}{4\pi\varepsilon_0 x} = \frac{x^2\rho}{3\varepsilon_0}$$

0.2

and will therefore gain electrical potential energy $\left(\frac{x^2\rho}{3\varepsilon_0}\right)\left(4\pi r^2\rho\right)\delta x$

0.1

Total energy of complete sphere = $\int_{x=0}^{x=R} \frac{4\pi\rho^2 x^4}{3\varepsilon_0} dx = \frac{4}{15}\frac{\pi\rho^2 R^5}{\varepsilon_0}$

0.2

Putting Q = charge on sphere = $\frac{4}{3}\pi R^3 \rho$, $\rho = \frac{3Q}{4\pi R^3}$

So that total energy is = $\frac{4}{15}\pi\left(\frac{9Q^2}{16\pi^2 R^6}\right)\frac{R^5}{\varepsilon} = \frac{3}{20}\frac{Q^2}{\pi\varepsilon_0 R}$

0.2

0.8

(c) Binding energy $E_{\text{binding}} = E_{\text{electric}} - E_{\text{nuclear}}$

0.1

Binding energy is a negative energy

Therefore -8.768 = E_{electric} - 10.980 MeV per nucleon

E_{electric} = 2.212 MeV per nucleon

0.1

Radius of cobalt nucleus is given by $R = \frac{3}{20}\frac{Q^2}{\pi\varepsilon_0 E_{electric}^{total}}$

$$= \frac{3\times 27^2 \times (1.60\times 10^{-19})^2}{20\times\pi\times 8.85\times 10^{-12}\times 2.212\times 10^6 \times 57\times 1.60\times 10^{-19}} \text{ m}$$

$= 5.0 \times 10^{-15}$ m

0.2

0.4

E E.M. Induction

Method 1 Equating energy

Horizontal component of magnetic field B inducing emf in ring:

$B = 44.5 \times 10^{-6} \cos 64^{o}$ 0.2

Magnetic flux through ring at angle $\theta = B\pi a^2 \sin \theta$

where a = radius of ring 0.1

Instantaneous emf = $\frac{\mathrm{d}\phi}{\mathrm{d}t} = B\pi a^2 \frac{\mathrm{d}\sin \omega t}{\mathrm{d}t}$ where ω = angular velocity

$= B\pi a^2 \omega \cos \omega t = B\pi a^2 \omega \cos \theta$ 0.1

R.m.s. emf over 1 revolution = $\frac{B\pi a^2 \omega}{\sqrt{2}}$ ***0.2***

Average resistive heating of ring = $\frac{B^2 \pi^2 a^4 \omega^2}{2R}$ 0.1

Moment of inertia = $\frac{1}{2} ma^2$ 0.1

Rotational energy = $\frac{1}{4} ma^2 \omega^2$ where m = mass of ring 0.1

Power producing change in ω = $\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{4} ma^2 \omega^2\right\}$ =

$\frac{1}{4} ma^2 2\omega \frac{\mathrm{d}\omega}{\mathrm{d}t}$

0.1

Equating: $\frac{1}{2} ma^2 \omega \frac{\mathrm{d}\omega}{\mathrm{d}t} = -\frac{B^2 \pi^2 a^4 \omega^2}{2R}$

0.1

$$\frac{\mathrm{d}\omega}{\omega} = -\frac{B^2 \pi^2 a^2}{mR} \mathrm{d}t$$

0.1

If T is time for angular velocity to halve,

$$\int_{\omega}^{\omega/2} \frac{\mathrm{d}\omega}{\omega} = -\int_0^T \frac{B^2 \pi^2 a^2}{mR} \mathrm{d}t$$

0.1

$$\ln 2 = \frac{B^2 \pi^2 a^2}{mR} T$$ 0.2

But R = $\frac{2\pi a\rho}{\mathrm{A}}$ where A is cross-sectional area of copper ring 0.1

$m = 2\pi ad A$ (d = density) 0.1

$$\ln 2 = \frac{B^2 \pi^2 a^2 T}{\frac{2\pi a\rho}{A} 2\pi adA} = \frac{B^2 T}{4\rho d}$$ 0.1

$$T = \frac{4\rho d \ln 2}{B^2} = \frac{4\times 1.70\times 10^{-8}\times 8.90\times 10^{3}\times 0.6931}{(44.5\times 10^{-6}\times 0.4384)^2}\text{s}$$

$= 1.10(2) \times 10^6$ s (=306 hr = 12 days 18 hr) 0.2

2.0

(Part E)

Method 2 Back Torque

Horizontal component of magnetic field = B = $44.5 \times 10^{-6} \cos 64^o$ 0.2

Cross-section of area of ring is A

Radius of ring = a

Density of ring = d

Resistivity = ρ

ω = angular velocity (ω positive when clockwise)

Resistance $R = \rho \dfrac{2\pi a}{A}$ 0.1

Mass of ring m = $2\pi a A d$ 0.1

Moment of inertia = $M = \dfrac{1}{2} m a^2$ 0.1

Magnetic flux through ring at angle $\theta = B\pi a^2 \sin\theta$ 0.1

Instantaneous emf = $\dfrac{d\phi}{dt} = B\pi a^2 \dfrac{d \sin \omega t}{dt} = B\pi a^2 \omega \cos \omega t = B\pi a^2 \omega \cos\theta$ 0.1

Induced current = I = $B\pi a^2 \cos\theta / R$

Torque opposing motion = $(B\pi a^2 \cos\theta)\, I = \dfrac{1}{R}\left(B\pi\, a^2\right)^2 \omega \cos^2\theta$ 0.1

Work done in small $\delta\theta = \dfrac{1}{R}\left(B\pi\, a^2\right)^2 \omega \dfrac{1}{2}\left(\cos 2\theta + 1\right)\delta\theta$ 0.1

Average torque = (work done in 2π revolution)/2π

$$= \frac{1}{2\pi R}\left(B\pi\pi^{\,2}\right)^2 \omega \frac{1}{2} 2\pi = \frac{1}{2R}\left(B\pi\pi^{\,2}\right)^2 \omega$$ 0.1

This equals $M\dfrac{d\omega}{dt}$ so that $M\dfrac{d\omega}{dt} = -\dfrac{B(\pi a^2)B(\pi a^2)\frac{1}{2}}{(\rho/A)(2\pi a)}\omega$ 0.2

$$\frac{1}{2}(2\pi a A d)a^2 \frac{d\omega}{dt} = -\frac{B^2(\pi a^2)^2 A}{4\rho\pi a}\omega$$

$$\frac{d\omega}{dt} = -\frac{B^2}{4\rho d}\omega$$ 0.2

$$\int_{\omega}^{\omega/2} \frac{d\omega}{\omega} = \int_0^T \frac{B^2}{4\rho d} dt$$ 0.2

$$\ln 2 = \frac{B^2 T}{4\rho d}$$ 0.2

$$T = \frac{4\rho d \ln 2}{B^2} = \frac{4 \times 1.70 \times 10^{-8} \times 8.90 \times 10^3 \times 0.6931}{(44.5 \times 10^{-6} \times 0.4384)^2} \text{s}$$

$= 1.10(2) \times 10^6$ s =306 hr = 12 days 18 hr

0.2

2.0

Theoretical Problem 2

(a) A cathode ray tube (CRT), consisting of an electron gun and a screen, is placed within a uniform constant magnetic field of magnitude **B** such that the magnetic field is parallel to the beam axis of the gun, as shown in figure 2.3.

The electron beam emerges from the anode of the electron gun on the axis, but with a divergence of up to 5° from the axis, as illustrated in figure 2.4. In general a diffuse spot is produced on the screen, but for certain values of the magnetic field a sharply focused spot is obtained.

By considering the motion of an electron initially moving at an angle β (where $0 \leq \beta \leq 5°$) to the axis as it leaves the electron gun, and considering the components of its motion parallel and perpendicular to the axis, derive an expression for the charge to mass ratio e/m for the electron in terms of the following quantities:

the smallest magnetic field for which a focused spot is obtained,
the accelerating potential difference across the electron gun V
(note that $V < 2$ kV),
and D, the distance between the anode and the screen.

Write your expression in the box provided in section 2a of the answer sheet.

(b) Consider another method of evaluating the charge to mass ratio of the electron. The arrangement is shown from a side view and in plan view (from above) in figure 2.5, with the direction of the magnetic field marked **B**. Within this uniform magnetic field **B** are placed two brass circular plates of radius ρ which are separated by a very small distance t. A potential difference V is maintained between them. The plates are mutually parallel and co-axial, however their axis is perpendicular to the magnetic field. A photographic film, covers the inside of the curved surface of a cylinder of radius ρ + s, which is held co-axial with the plates. In other words, the film is at a radial distance s from the edges of the plates. The entire arrangement is placed *in vacuo*. Note that t is very much smaller than both s and ρ.

A point source of β particles, which emits the β particles uniformly in all directions with a range of velocities, is placed between the centres of the plates, and the *same piece of film* is exposed under three different conditions:

firstly with $B = 0$, and $V = 0$,
secondly with $B = B_0$, and $V = V_0$, and
thirdly with $B = -B_0$, and $V = -V_0$;

where V_0 and B_0 are positive constants. Please note that the upper plate is positively charged when $V>0$ (negative when $V<0$), and that the magnetic field is in the direction defined by figure 2.5 when $B>0$ (in the opposite direction when $B<0$). For this part you may assume the gap is negligibly small.

Two regions of the film are labelled A and B on figure 2.5. After exposure and development, a sketch of one of these regions is given in figure 2.6. From which region was this piece taken (on your answer sheet write A or B)? Justify your answer by showing the directions of the forces acting on the electron.

(c) After exposure and development, a sketch of the film is given in figure 2.6. Measurements are made of the separation of the two outermost traces with a microscope, and this distance (y) is also indicated for one particular angle on figure 2.6. The results are given in the table below, the angle ϕ being defined in figure 2.5 as the angle between the magnetic field and a line joining the centre of the plates to the point on the film.

Angle to field /degrees	ϕ	90	60	50	40	30	23
Separation /mm	y	17.4	12.7	9.7	6.4	3.3	End of trace

Numerical values of the system parameters are given below:

$B_0 = 6.91$ mT $\quad V_0 = 580$ V $\quad t = 0.80$ mm

$s = 41.0$ mm

In addition, you may take the speed of light in vacuum to be 3.00×10^8 m s^{-1}, and the rest mass of the electron to be 9.11×10^{-31} kg.

Determine the maximum β particle kinetic energy observed.

Write the maximum kinetic energy as a numerical result in eV in the box on the answer sheet, section 2f.

(d) Using the information given in part (f), obtain a value for the charge to rest mass ratio of the electron. This should be done by plotting an appropriate graph on the paper provided.

Indicate *algebraically* the quantities being plotted on the horizontal and vertical axes both on the graph itself *and* on the answer sheet in the boxes provided in section 2g.

Write your value for the charge to mass ratio of the electron in the box provided on the answer sheet, section 2g.

Please note that the answer you obtain may not agree with the accepted value because of a systematic error in the observations.

Question Two ~ Solution

(a) Focusing occurs for one "cyclotron" orbit of the electron.

Angular velocity $\omega = \mathrm{e}\,B / \mathrm{m}$; so time for one orbit $T = 2\,\pi\,\mathrm{m} / \mathrm{e}\,B$

Speed of electron $u = (2\,\mathrm{e}\,V / \mathrm{m})^{1/2}$

Distance travelled $D = T\,u \cos\beta \approx T\,u = (2^{3/2}\,\pi / B)\,(V\,\mathrm{m} / \mathrm{e})^{1/2}$

Thus charge to mass ratio = $\mathrm{e} / \mathrm{m} = 8\,\mathrm{V} \times (\pi / B\,D)^2$

(b) Consider condition (ii) - Force due to electric field acts upwards

In region A force due magnetic field acts upwards as well, electron hits upper plate and does not reach the film.

In region B, force due magnetic field acts downwards, and *if* force is equal and opposite to the electrostatic force, there will be no unbalanced force, and electron will emerge from plates to expose film.

Piece was taken from region B.

(c) We require forces to balance. Electric force given by $\mathrm{e}V / t$, magnitude of magnetic force given by $\mathrm{e}\,u\,B \sin\phi$, with u the speed of the electron.

For these to balance we require $u = V / t\,B\,|\sin\phi|$

Maximum u corresponds to minimum ϕ - at 23°

Therefore $u = 2.687 \times 10^8$ m/s = 0.896 c.

Relativistic $\gamma = (1 - v^2/\mathrm{c}^2)^{-1/2} = 2.255$,
so kinetic energy of electron = $(\gamma - 1)\,\mathrm{m}\,\mathrm{c}^2 = 641$ keV.

(d) After emerging from region between plates, electrons experience force due to magnetic field only. We approximate this by a vertical force, because angle of electron to horizontal remains small.

Acceleration caused by this force $a = B\ \mathrm{e}\ u \sin\phi / \gamma \mathrm{m}$

Initial horizontal speed is u, therefore time taken to reach the film after emerging from the region between the plates $t = s / u$.

Change in vertical displacement during this time $= y / 2 = \frac{1}{2} a (s / u)^2$

$y = B\ \mathrm{e}\ s^2 \sin\phi / \gamma \mathrm{m}\ u$

From part (f), for electron to have emerged from plate,
we also know $u = V / t B |\sin\phi|$.

Therefore we eliminate u to obtain:

$$y^2 = (\mathrm{e} B s \sin\phi / \mathrm{m})^2 \ \{(B s t \sin\phi / V)^2 - (s / \mathrm{c})^2\}$$

and we plot	VERTICAL	$(y / B s \sin\phi)^2$
	HORIZONTAL	$(B s t \sin\phi / V)^2$

Therefore we have a gradient $(\mathrm{e} / \mathrm{m})^2$

and a vertical-axis intercept $-(\mathrm{e}\ s / \mathrm{m\ c})^2$

The intercept is read as -537.7 $(\mathrm{C\ s / kg})^2$, giving $\mathrm{e/m} = 1.70 \times 10^{11}$ C / kg

The gradient is read as 2.826×10^{22} $(\mathrm{C/kg})^2$, giving $\mathrm{e/m} = 1.68 \times 10^{11}$ C / kg.

Question Two ~ Mark Scheme

(a)	Focus after integer number of cyclotron revolutions	0.6
	For smallest current, we have only one revolution	0.3
	$\omega = e B / m$, and hence $T = 2 \pi m / e B$	0.4
	Neglect of cos β factor and justification; hence $D = T u$	0.2
	Speed $u = (2 V e / m)^{1/2}$	0.1
	Final expression dimensionally correct	0.2
	in addition: final answer correct apart from a numerical factor	0.3
	in addition: answer fully correct	0.6
	Justification of neglect of relativity	0.3
	TOTAL	**3.0**
(b)	Electrons emerge if electric and magnetic forces equal and opposite	0.4
	Correct direction for electric force (both regions)	0.2
	Correct direction for magnetic force (region A) 0.3	
	Correct direction for magnetic force (region B) 0.3	
	Correct region chosen	0.3
	TOTAL	**1.5**
(c)	Correct magnitude for electric and magnetic forces	0.4
	Maximum speed corresponds to minimum ϕ	0.3
	Correct expression for speed	0.1
	Correct value for speed	0.3
	Note of relativistic magnitude of speed	0.2
	Correct use and definition of γ	0.3
	Correct energy (in correct units)	0.4
	TOTAL	**2.0**
(d)	Use of expression for speed u from part (f)	0.1
	Assumption that direction of force remains constant (or alternative)	0.3
	Relationship between y and unbalanced magnetic force	0.1
	Relativistic mass consideration	0.3
	$y = B e s^2 \sin \phi / (\gamma m u)$	0.3
	Correct equation with speed eliminated	0.3
	Sensible choice of what to plot	0.6
	Clear, well labelled and accurately plotted graph	0.6
	Gradient or intercept correctly read and calculated from graph	0.3
	Correct value for e/m (in correct units)	0.6
	TOTAL	**3.5**

GRAND TOTAL 10.0

Theoretical Problem 3

Gravitational waves and the effects of gravity on light.

Part A

This part is concerned with the difficulties of detecting gravitational waves generated by astronomical events. It should be realised that the explosion of a distant supernova may produce fluctuations in the gravitational field strength at the surface of the Earth of about 10^{-19} N kg^{-1}.

A model for a gravitational wave detector (see figure 3.1) consists of two metal rods each 1m long, held at right angles to each other. One end of each rod is polished optically flat and the other end is held rigidly. The position of one rod is adjusted so there is a minimum signal received from the photocell (see figure 3.1).

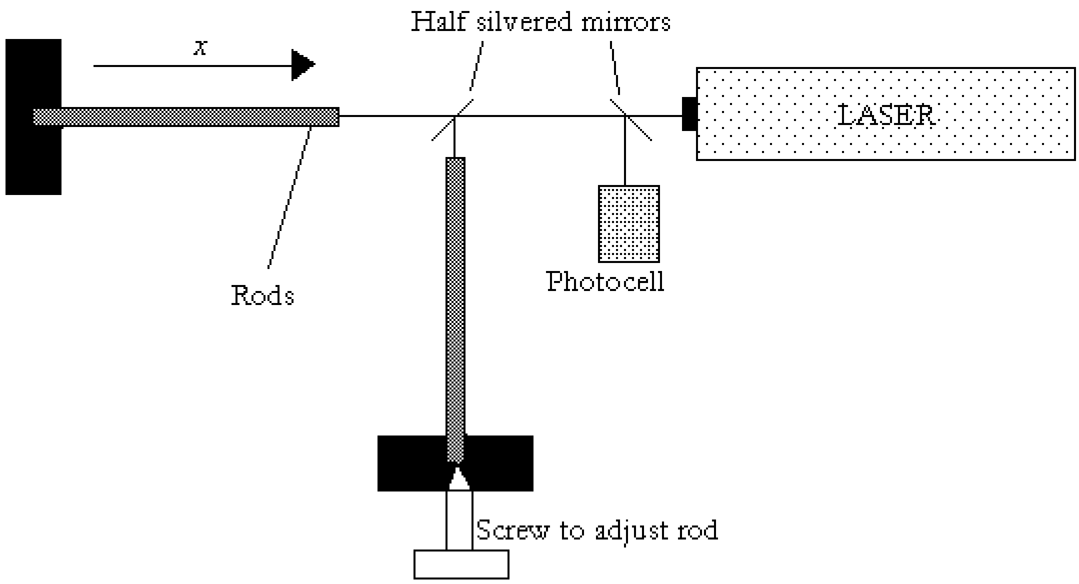


Figure 3.1

The rods are given a short sharp impulse by a piezoelectric device. As a result the free ends of the rods oscillate with a longitudinal displacement Δx_t, where

$$\Delta x_t = ae^{-\mu t}\cos(\omega t + \phi),$$

and a, μ, ω and ϕ are constants.

(a) If the amplitude of the motion is reduced by 20% during a 50s interval determine a value for μ.

(b) Determine also a value for ω given that the rods are made of aluminium with a density (ρ) of 2700 kg m^{-3} and a Young modulus (E) of 7.1×10^{10} Pa.

(c) It is impossible to make the rods exactly the same length so the photocell signal has a beat frequency of 0.005 Hz. What is the difference in length of the rods?

(d) For the rod of length l, derive an algebraic expression for the change in length, Δl, due to a change, Δg, in the gravitational field strength, g, in terms of l and other constants of the rod material.

(e) The light produced by the laser is monochromatic with a wavelength of 656nm. If the minimum fringe shift that can be detected is 10^{-4} of the wavelength of the laser, what is the minimum value of l necessary if such a system were to be capable of detecting variations in g of 10^{-19} N kg^{-1}?

A non-directional form of gravitational wave detector consists of a sphere of copper-alloy of mass 1168 kg, suspended in a vacuum from a vibration-reducing assembly. Transducers, containing tuned circuits, are attached to the sphere to detect changes in its dimensions. The transducers will, however, pick up all spurious vibrations due to, for example, temperature effects and noise due to electric pick up.

(f) To reduce vibrations due to temperature effects the sphere is maintained at a temperature of 100 mK. By what factor will the amplitude of the atomic vibrations been reduced in cooling the assembly from 300 K?

(g) The sphere is initially cooled to 4.2 K using liquid nitrogen and liquid helium. The temperature, T, is further reduced to 100 mK by a refrigeration process, which removes energy from the system at an average rate of 1 mW. Given that the specific thermal capacity, s, of the copper-alloy varies directly as T^3 at these low temperatures, estimate the time taken for the system to cool from 4.2 K to 100 mK, given that s = 0.072 J kg^{-1} K^{-1} at 4.2 K.

Part B

This part is concerned with the effect of a gravitational field on the propagation of light in space.

(a) A photon emitted from the surface of the Sun (mass M, radius R) is red-shifted. By assuming a rest-mass equivalent for the photon energy, apply Newtonian gravitational theory to show that the effective (or measured) frequency of the photon at infinity is reduced (red-shifted) by the factor $(1 - GM/Rc^2)$.

(b) A reduction of the photon's frequency is equivalent to an increase in its time period, or, using the photon as a standard clock, a dilation of time. In addition, it may be shown that a time dilation is always accompanied by a contraction in the unit of length by the same factor.

We will now try to study the effect that this has on the propagation of light near the Sun. Let us first define an effective refractive index n_r at a point r from the centre of the Sun. Let

$$n_r = \frac{c}{c_r'},$$

where c is the speed of light as measured by a co-ordinate system far away from the Sun's gravitational influence ($r \to \infty$), and c_r' is the speed of light as measured by a co-ordinate system at a distance r from the centre of the Sun.

Show that n_r may be approximated to:

$$n_r = 1 + \frac{\alpha GM}{rc^2},$$

for small **GM/rc^2*****, where*** α ***is a constant that you determine.***

(c) Using this expression for n_r, calculate in radians the deflection of a light ray from its straight path as it passes the edge of the Sun.

Data:

Gravitational constant, $G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}.
Mass of Sun, $M = 1.99 \times 10^{30}$ kg.
Radius of Sun, $R = 6.95 \times 10^{8}$ m.
Velocity of light, $c = 3.00 \times 10^{8}$ m s^{-1}.

You may also need the following integral:

$$\int_{-\infty}^{+\infty} \frac{dx}{(x^2 + a^2)^{3/2}} = \frac{2}{a^2}.$$

MARK SCHEME AND SOLUTIONS FOR Q3

Total marks = 10

A **a)** $\Delta x_t = ae^{-\mu t}\cos(\omega t + \phi)$, $0.8 = e^{-50\mu} \Rightarrow \mu = 4.5 \times 10^{-3}\ s^{-1}$. [0.1]

b) $\upsilon = (E/\rho)^{½} = (7.1 \times 10^{10}/2700)^{½} = 5100\ m.s^{-1}$.
At fundamental $\lambda_{rod} = 4l = 4$ m.
$f = 5100 / 4 = 1.3 \times 10^3$ Hz.
$\omega = 2\pi f = 8.1 \times 10^3\ rad.s^{-1}$. [0.1]

c) $\upsilon = f\lambda_{rod}$, $\delta\lambda_{rod} / \lambda_{rod} = (-)\delta f / f \Rightarrow \delta l / l$. [0.8]
$\delta l = l.(\delta f / f)$. [0.6]
$\delta l = 1 \times (5.0 \times 10^{-3}/1.3 \times 10^3) = 3.8 \times 10^{-6}$ m. [0.1]

d) Change in gravitational force on rod at a distance x from the free end = $m\Delta g$ and $m = \rho x A$,
where A is the cross-sectional area of the rod. [0.5]
Change in stress = $m\Delta g/A = \rho x\Delta g$. [0.5]
Change in strain = $\delta(dx)/dx = \rho x \Delta g/E$;
that is, $dx \rightarrow (1 + \rho x\Delta g/E)dx \Rightarrow \Delta l = (\rho\Delta g/2E)l^2$. [0.5]

e) At fundamental $\lambda_{rod} = 4l \Rightarrow \Delta l = \Delta\lambda_{rod}/4$,
for $\Delta\lambda_{rod} = 656\ nm/10^4 \Rightarrow \Delta l = 656\ nm/(4 \times 10^4)$. [0.1]
$\Delta l = 656\ nm/(4 \times 10^4) = (\rho\Delta g/2E)l^2$ [0.1]
$\Delta l = (2700 \times 10^{-19}/ 14 \times 10^{10})\ l^2 \Rightarrow l = 9.2 \times 10^7$ m. [0.1]

B **a)** $mc^2 = hf \Rightarrow m = hf/c^2$, [0.3]
$hf' = hf - GMm/R$, [0.3]
$\Rightarrow hf' = hf(1 - GM/Rc^2)$, $\therefore f' = f(1 - GM/Rc^2)$. [0.4]

b) $n_r = c / c(1 - GM/rc^2)^2$, [1.0]
$n_r = 1 + 2GM/rc^2$, for small GM/rc^2; i.e. $\alpha = 2$. [1.0]

c)

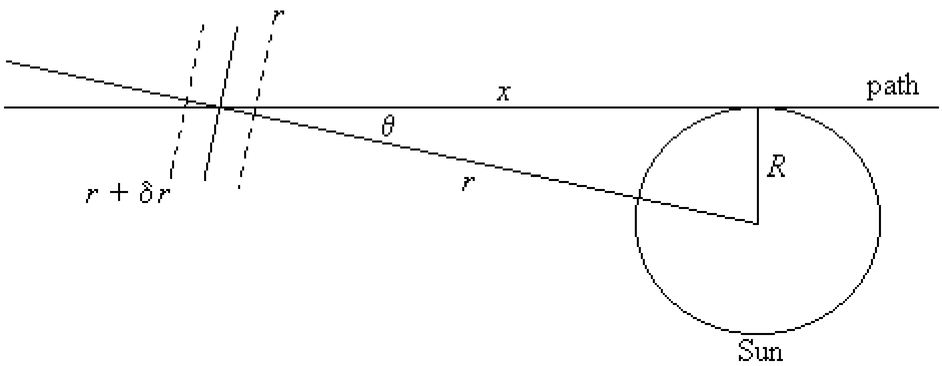


Diagram [0.2]

By Snell's law: $n(r + \delta r) \sin \theta = n(r) \sin (\theta - \delta\xi)$, [1.0]

$(n(r) + (dn/dr)\, \delta r) \sin \theta = n(r) \sin \theta - n(r) \cos \theta\, \delta\xi$. [0.4]

$(dn/dr)\, \delta r \sin \theta = -\, n(r) \cos \theta\, \delta\xi$.

Now $n(r) = 1 + 2GM/rc^2$, so $(dn/dr) = -\, 2GM/c^2r^2$, [0.3]

and $(2GM/c^2r^2) \sin \theta\, \delta r = n(r) \cos \theta\, \delta\xi$.

Hence $\delta\xi = (2GM/c^2r^2) \tan \theta\, (\delta r/n) \approx (2GM \tan \theta /c^2r^2)\delta r$. [1.0]

Now $r^2 = x^2 + R^2$, so $rdr = xdx$. [0.1]

$$\int d\xi = \frac{2GM}{c^2}\int \frac{\tan\theta dr}{r^2} = \frac{2GM}{c^2}\int \frac{\tan\theta r dr}{r^3} = \frac{2GMR}{c^2}\int_{-\infty}^{+\infty} \frac{dx}{(x^2 + R^2)^{3/2}}$$

$$\xi = \frac{4GM}{Rc^2} radians = 8.4 \times 10^{-6} radians.$$

[0.5]