Conic Sections and Polar Coordinates

10.6 Graphing

10.7 Areas and Lengths

10.8 Conic Sections

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Polar to Rectangular Coordinates:

$$x = r\cos\theta, \qquad y = r\sin\theta\,. \tag{1}$$

If $r = 0$ and $\theta \in \mathbb{R}$ the described point $P(x, y)$ is the **origin** $(0, 0)$. If any **other** point $P(x, y)$ is described by polar coordinates (r, θ) or (r', θ') then these coordinates are related by

$$r\cos\theta = r'\cos\theta' \quad \text{and} \quad r\sin\theta = r'\sin\theta'$$

or equivalently

$$\begin{aligned} r' &= r \quad \text{and} \quad \theta' = \theta + (2n)\pi, \; n \in \mathbb{Z} \\ &\text{or} \\ r' &= -r \quad \text{and} \quad \theta' = \theta + (2n+1)\pi, \; n \in \mathbb{Z}\,. \end{aligned}$$

Rectangular to Polar Coordinates:

For a point $P(x, y)$ different from the origin a polar coordinate description is given by

$$r = \sqrt{x^2 + y^2},\ \theta = \tan^{-1}\frac{y}{x} \text{ if } x \neq 0 \text{ or } \theta = \cot^{-1}\frac{x}{y} \text{ if } y \neq 0\,. \quad (2)$$

Symmetry: Reflection in x-axis

$$
\begin{aligned}
(x, y) &\mapsto (x, -y) \\
&\Leftrightarrow \\
(r\cos(\theta), r\sin(\theta)) &\mapsto (r\cos(\theta), -r\sin(\theta)) \\
&= (r\cos(-\theta), r\sin(-\theta)) \\
&= (-r\cos(\pi - \theta), -r\sin(\pi - \theta)) \\
&\Leftrightarrow \\
(r, \theta) &\mapsto (r, -\theta) \\
&\text{or} \\
(r, \theta) &\mapsto (-r, \pi - \theta).
\end{aligned}
$$

Symmetry: Reflection in y-axis

$$
\begin{aligned}
(x, y) &\mapsto (-x, y) \\
&\Leftrightarrow \\
(r\cos(\theta), r\sin(\theta)) &\mapsto (-r\cos(\theta), r\sin(\theta)) \\
&= (r\cos(\pi-\theta), r\sin(\pi-\theta)) \\
&= (-r\cos(-\theta), -r\sin(-\theta)) \\
&\Leftrightarrow \\
(r, \theta) &\mapsto (r, \pi-\theta) \\
&\text{or} \\
(r, \theta) &\mapsto (-r, -\theta)\,.
\end{aligned}
$$

Symmetry: Reflection in origin:

$$
\begin{aligned}
(x,y) &\mapsto (-x,-y) \\
&\Leftrightarrow \\
(r\cos(\theta), r\sin(\theta)) &\mapsto (-r\cos(\theta), -r\sin(\theta)) \\
&= (r\cos(\theta+\pi), r\sin(\theta+\pi)) \\
&\Leftrightarrow \\
(r,\theta) &\mapsto (-r,\theta) \\
&\text{or} \\
(r,\theta) &\mapsto (r,\theta+\pi)\,.
\end{aligned}
$$

Slope of the Curve $r = f(\theta)$:

Given a function $r = f(\theta)$ the equations

$$x = f(\theta)\cos\theta \quad \text{and} \quad y = f(\theta)\sin\theta \tag{3}$$

provide a parameterized description of a curve in the (x, y)-plane.

Assuming $f'(\theta)\cos\theta - f(\theta)\sin\theta \neq 0$, the slope of the tangent to this parameterized curve can be computed by

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}\,.$$

When $f(\theta) = 0$ the slope equals $\frac{dy}{dx} = \tan\theta$, while $f'(\theta) = 0$ implies $\frac{dy}{dx} = -\cot\theta$, so that in that case the tangent is perpendicular to the ray from the origin to the point $P(x, y)$.

Area in the Plane:

The area described by the conditions

$\alpha \le \theta \le \beta,\ 0 \le r \le f(\theta)$ is given by

$$A = \int_{\alpha}^{\beta} \frac{1}{2}(f(\theta))^2 \, d\theta \,. \tag{4}$$

The area described by the conditions

$\alpha \le \theta \le \beta,\ 0 \le f_1(\theta) \le r \le f_2(\theta)$ is given by

$$A = \int_{\alpha}^{\beta} \frac{1}{2}\left((f_2(\theta))^2 - (f_1(\theta))^2\right) \, d\theta \,. \tag{5}$$

Length of a Polar Curve:

Assuming that $r = f(\theta)$ is continuously differentiable for $\alpha \leq \theta \leq \beta$ and that the point $P_{polar}(r, \theta)$ traces the graph exactly once, the length of the curve is given as follows

$$\begin{aligned} dx &= \Big(f'(\theta)\cos\theta - f(\theta)\sin\theta\Big)\, d\theta \\ dy &= \Big(f'(\theta)\sin\theta + f(\theta)\cos\theta\Big)\, d\theta \\ ds^2 = dx^2 + dy^2 &= \Big(f'^2(\theta) + f^2(\theta)\Big)\, d\theta^2 \end{aligned}$$

so that

$$L = \int ds = \int_\alpha^\beta \sqrt{f'^2(\theta) + f^2(\theta)}\, d\theta\,. \qquad (6)$$

Area of Surface of Revolution of a Polar Curve

Assuming that $r = f(\theta)$ is continuously differentiable for $\alpha \leq \theta \leq \beta$ and that the point $P_{polar}(r, \theta)$ traces the graph exactly once, the areas of the surfaces generated by revolving the curve around the x- and y-axes is given as follows

$$x\text{-axis:} \quad S = \int_{\alpha}^{\beta} 2\pi f(\theta) \sin\theta \sqrt{f'^2(\theta) + f^2(\theta)}\, d\theta \qquad (7)$$

$$y\text{-axis:} \quad S = \int_{\alpha}^{\beta} 2\pi f(\theta) \cos\theta \sqrt{f'^2(\theta) + f^2(\theta)}\, d\theta \qquad (8)$$

Polar Equations for Lines:

If the perpendicular to a **line** L from the origin meets the line at the point $P_{polar}(r_0, \theta_0)$, where $r_0 > 0$ then the general point $P_{polar}(r, \theta)$ of the line L satisfies the equation

$$r\cos(\theta - \theta_0) = r_0\,. \qquad (9)$$

Polar Equations for Circles:

The general point $P_{polar}(r, \theta)$ of the **circle** with center $P_{polar}(r_0, \theta_0)$ and radius $a > 0$ satisfies the equation

$$a^2 \;=\; r^2 + r_0^2 - 2rr_0\cos(\theta - \theta_0)\,. \qquad (10)$$

Ellipses, Parabolas, and Hyperbolas:

Using the eccentricity $e > 0$ in the focus-directrix definition of the conic sections, where the focus F is assumed at the origin and the directrix D is described by $x = k$ for some $k > 0$, one finds the polar equations

$$\begin{aligned} PF &= e\,PD \\ r &= e(k - x) \\ &= e(k - r\cos\theta) \\ r(1 + e\cos\theta) &= ek \quad \text{so that, finally} \end{aligned}$$

$$r = \frac{ek}{1 + e\cos\theta}\,. \tag{11}$$

Standard Ellipse:

An ellipse (not a circle) has an eccentricity e with $0 < e < 1$.
According to Figure 10.19 a translation in the x-direction by $-c$ verifies that $k = \frac{a}{e} - c$ so that $ek = a - ec = \left(1 - e^2\right) a$ and

$$r = \frac{\left(1 - e^2\right) a}{1 + e \cos \theta}. \tag{12}$$

Standard Parabola:

A parabola has eccentricity $e = 1$.

It is seen that $k = 2p > 0$ so that the equation for the parabola opening up to the left and with axis equal to the x-axis is

$$r = \frac{2p}{1 + \cos\theta}\,. \tag{13}$$

Standard Hyperbola:

A hyperbola has an eccentricity e with $e > 1$.

According to Figure 10.20 a translation in the x-direction by c verifies that $k = c - \frac{a}{e}$ so that $ek = ec - a = \left(e^2 - 1\right) a$ and

$$r = \frac{\left(e^2 - 1\right) a}{1 + e \cos\theta}, \quad \text{with } |\theta| < \cos^{-1}\left(-\frac{1}{e}\right). \tag{14}$$